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AVERAGING OF THE HEAT-TRANSFER COEFFICIENT IN THE PROCESSES OF HEAT EXCHANGE WITH PERIODIC INTENSITY

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This work presents a proof of the inequality that determines the relation between two values of the heat-transfer coefficient averaged by different procedures in processes of heat exchange with periodic intensity.

In [1], an approximate method is suggested to investigate a conjugate "heat carrier-wall" problem for heat-exchange processes whose intensity changes periodically along the heat-exchange surface (along the coordinate z) and in time τ. We consider a boundary-value problem for the two-dimensional nonstationary equation of heat conduction in a wall (a plate of thickness δ) with the third-kind boundary condition at $x = \delta$

$$\alpha = \frac{q_\delta}{\vartheta_\delta} \tag{1}$$

and the appropriate boundary condition at $x = 0$. All the quantities in Eq. (1) are represented in the form of superposition of the averaged and periodic components

$$q_\delta = \langle q_\delta \rangle (1 + \tilde{q}_\delta) , \quad \vartheta_\delta = \langle \vartheta_\delta \rangle (1 + \tilde{\vartheta}_\delta) , \quad \alpha = \langle \alpha \rangle (1 + \tilde{\alpha}) . \tag{2}$$

Substitution of Eq. (2) into Eq. (1) and averaging yield

$$\langle \alpha \rangle = \left\langle \frac{q_\delta}{\vartheta_\delta} \right\rangle = \frac{\langle q_\delta \rangle}{\langle \vartheta_\delta \rangle} \left\langle \frac{1 + \tilde{q}_\delta}{1 + \tilde{\vartheta}_\delta} \right\rangle . \tag{3}$$

Introducing the notation

$$\alpha_{\rm m} = \frac{\langle q_\delta \rangle}{\langle \vartheta_\delta \rangle} \tag{4}$$

and taking into account that the quantity $\alpha_{\rm m}$ is measured in traditional heat-exchange experiments and is used in applied calculations, we call it the "experimental heat-transfer coefficient." The quantity $\langle \alpha \rangle$, prescribed a priori in boundary condition (1), is called the "averaged true heat-transfer coefficient."

The difference in the values of the heat-transfer coefficient averaged by procedures (3) and (4) can be conveniently characterized as the "conjugation parameter":

$$\varepsilon = \frac{\alpha_{\rm m}}{\langle \alpha \rangle} . \tag{5}$$

In [1], for a number of particular cases the double inequality below was proved

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$$\left\langle \frac{1}{1+\tilde{\alpha}} \right\rangle^{-1} \le \varepsilon \le 1 , \tag{6}$$

from which it follows that the experimental heat-transfer coefficient is smaller than the averaged true heat-transfer coefficient (the sign of equality is attained in the limit of an infinitely heat-conducting wall).

In [2-6], in analyzing practical applications of the method of [1], we used the conjugation parameter ε as a correction factor that accounted for the entire complex of the influence of a solid wall on the quantity α_m (the thermophysical properties and thickness of the wall, the method of supplying heat to the solid body and its geometry, the amplitude, the spatial and time periods of pulsations): $\alpha_m = \varepsilon\langle\alpha\rangle$. The objective of the present work is to prove in general form double inequality (6) that determines the range of variation in the conjugation parameter ε.

Proof of the inequality $\varepsilon \le 1$. First, we prove the auxiliary inequality

$$\langle \tilde{\vartheta}_\delta \tilde{q}_\delta \rangle \le 0 . \tag{7}$$

A heat-conduction equation for a periodic component of the temperature field with allowance for the Fourier law

$$\tilde{q}_x = -\frac{\lambda}{\alpha_m}\frac{\partial\tilde{\vartheta}}{\partial x}, \quad \tilde{q}_z = -\frac{\lambda}{\alpha_m}\frac{\partial\tilde{\vartheta}}{\partial z} \tag{8}$$

can be written in the form

$$c\rho\frac{\partial\tilde{\vartheta}}{\partial\tau} = -\alpha_m\left(\frac{\partial\tilde{q}_x}{\partial x} + \frac{\partial\tilde{q}_z}{\partial z}\right). \tag{9}$$

Multiplying both sides of Eq. (9) by $\tilde{\vartheta}$, we obtain

$$\frac{c\rho}{2}\frac{\partial\tilde{\vartheta}^2}{\partial\tau} + \alpha_m\tilde{\vartheta}\left(\frac{\partial\tilde{q}_x}{\partial x} + \frac{\partial\tilde{q}_z}{\partial z}\right) = 0 . \tag{10}$$

Writing the identities

$$\tilde{\vartheta}\frac{\partial\tilde{q}_x}{\partial x} = \frac{\partial\tilde{\vartheta}\tilde{q}_x}{\partial x} - \tilde{q}_x\frac{\partial\tilde{\vartheta}}{\partial x}, \quad \tilde{\vartheta}\frac{\partial\tilde{q}_z}{\partial z} = \frac{\partial\tilde{\vartheta}\tilde{q}_z}{\partial z} - \tilde{q}_z\frac{\partial\tilde{\vartheta}}{\partial z}$$

and using Eq. (8), we rewrite Eq. (10) in the form

$$\frac{c\rho}{2\alpha_m}\frac{\partial\tilde{\vartheta}^2}{\partial\tau} + \frac{\partial\tilde{\vartheta}\tilde{q}_x}{\partial x} + \frac{\partial\tilde{\vartheta}\tilde{q}_z}{\partial z} + \frac{\alpha_m}{\lambda}(\tilde{q}_x^2 + \tilde{q}_z^2) = 0 . \tag{11}$$

Integration of both sides of Eq. (11) over x within the limits from 0 to δ yields

$$\frac{c\rho}{2\alpha_m}\frac{\partial}{\partial\tau}\int_0^\delta \tilde{\vartheta}^2 dx + \tilde{\vartheta}\tilde{q}_x\Big|_{x=0}^{x=\delta} + \frac{\partial}{\partial z}\int_0^\delta \tilde{\vartheta}\tilde{q}_z dx + \frac{\alpha_m}{\lambda}\int_0^\delta (\tilde{q}_x^2 + \tilde{q}_z^2)\, dx = 0 . \tag{12}$$

Denoting $\tilde{q}_0 \equiv (\tilde{q}_x)_{x=0}$ and $\tilde{q}_\delta \equiv (\tilde{q}_x)_{x=\delta}$, we rewrite Eq.(12) as

$$\tilde{\vartheta}_\delta\tilde{q}_\delta = \tilde{\vartheta}_0\tilde{q}_0 - \frac{c\rho}{2\alpha_m}\frac{\partial}{\partial\tau}\int_0^\delta \tilde{\vartheta}^2 dx - \frac{\partial}{\partial z}\int_0^\delta \tilde{\vartheta}\tilde{q}_z dx - \frac{\alpha_m}{\lambda}\int_0^\delta (\tilde{q}_x^2 + \tilde{q}_z^2)\, dx . \tag{13}$$

Then, averaging both sides of Eq. (13) over τ and z and noting that here the second and third terms on the right-hand side drop out, we obtain

$$\langle\tilde{\vartheta}_\delta\,\tilde{q}_\delta\rangle = \langle\tilde{\vartheta}_0\,\tilde{q}_0\rangle - \frac{\alpha_m}{\lambda}\int\limits_0^\delta \langle\tilde{q}_x^2 + \tilde{q}_z^2\rangle\,dx\,. \tag{14}$$

Now we show that for any type of boundary condition with $x = 0$ from Eq. (14) there follows the validity of auxiliary inequality (7).

1. Boundary conditions of the first and second kind: $\langle\tilde{\vartheta}\rangle$ = const, $\langle q_0\rangle$ = const. Here either $\tilde{\vartheta}_0 = 0$ or $\tilde{q}_0 = 0$, i.e., $\langle\tilde{\vartheta}_0\tilde{q}_0\rangle = 0$, whence

$$\langle\tilde{\vartheta}_\delta\,\tilde{q}_\delta\rangle = -\frac{\alpha_m}{\lambda}\int\limits_0^\delta \langle\tilde{q}_x^2 + \tilde{q}_z^2\rangle\,dx\,. \tag{15}$$

Since we always have $\langle\tilde{q}_x^2 + \tilde{q}_z^2\rangle \geq 0$, from Eq. (15) follows Eq. (7), which was to be proved.

2. The boundary condition of the third kind: α_0 = const. Here

$$\tilde{q}_0 = -\frac{\alpha_0}{\alpha_m}\,\tilde{\vartheta}_0\,. \tag{16}$$

Multiplying both sides of Eq. (16) by $\tilde{\vartheta}_0$ and averaging them, we obtain

$$\langle\tilde{\vartheta}_0\,\tilde{q}_0\rangle = -\frac{\alpha_0}{\alpha_m}\langle\tilde{\vartheta}_0^2\rangle\,. \tag{17}$$

Since $\tilde{\vartheta}_0^2 \geq 0$, from Eq. (17) we have

$$\langle\tilde{\vartheta}_0\,\tilde{q}_0\rangle \leq 0\,. \tag{18}$$

Equations (14) and (18) yield

$$\langle\tilde{\vartheta}_\delta\,\tilde{q}_\delta\rangle = -\left(\frac{\alpha_0}{\alpha_m}\langle\tilde{\vartheta}_0^2\rangle + \frac{\alpha_m}{\lambda}\int\limits_0^\delta \langle\tilde{q}_x^2 + \tilde{q}_z^2\rangle\,dx\right). \tag{19}$$

From Eq. (19) follows Eq. (7), which was to be proved.

3. The boundary condition of the "fourth kind." Suppose that the boundary condition with $x = 0$ is contact with the second plate, on the outer surface of which in turn the boundary condition of the first, second, or third kind is prescribed. Transforming the heat-conduction equation for the second plate by means of Eqs. (8)-(14), it is easy to obtain the proof of inequality (7). We do not present here the corresponding calculations because of their unwieldiness.

Proof of the inequality $\varepsilon \leq 1$. With account for Eqs. (2)-(5), we rewrite Eq. (1) in the form

$$\varepsilon\,(1 + \tilde{q}_\delta) = (1 + \tilde{\alpha})\,(1 + \tilde{\vartheta}_\delta)\,. \tag{20}$$

Averaging of Eq. (20) yields

$$\varepsilon = 1 + \langle\tilde{\alpha}\,\tilde{\vartheta}_\delta\rangle\,. \tag{21}$$

Multiplying both sides of Eq. (20) by $1 + \tilde{\vartheta}_\delta$ and averaging them, we obtain

$$\varepsilon\left(1+\langle\tilde{\vartheta}_\delta\tilde{q}_\delta\rangle\right)=1+2\langle\tilde{\alpha}\tilde{\vartheta}_\delta\rangle+\langle(1+\tilde{\alpha})\tilde{\vartheta}_\delta^2\rangle, \tag{22}$$

or, with account for Eq. (21),

$$\varepsilon=\frac{1-\langle(1+\tilde{\alpha})\tilde{\vartheta}_\delta^2\rangle}{1-\langle\tilde{\vartheta}_\delta\tilde{q}_\delta\rangle}. \tag{23}$$

Since $1+\tilde{\alpha}\geq 0$ and $\tilde{\vartheta}_\delta^2\geq 0$, we have $\langle(1+\tilde{\alpha})\tilde{\vartheta}_\delta^2\rangle\geq 0$. It follows that the condition needed for the inequality

$$\varepsilon\leq 1 \tag{24}$$

to be satisfied will be the satisfaction of inequality (7), which was proved above.

Proof of the inequality $\varepsilon\geq\left\langle\frac{1}{1+\tilde{\alpha}}\right\rangle^{-1}$. Having divided both sides of inequality (20) into the quantity $\varepsilon(1+\tilde{\alpha})$, we obtain

$$\varepsilon^{-1}(1+\tilde{\vartheta}_\delta)=\frac{1+\tilde{q}_\delta}{1+\tilde{\alpha}}. \tag{25}$$

Averaging of Eq. (25) yields

$$\varepsilon^{-1}=\left\langle\frac{1}{1+\tilde{\alpha}}\right\rangle+\left\langle\frac{\tilde{q}_\delta}{1+\tilde{\alpha}}\right\rangle. \tag{26}$$

Having divided both sides of Eq. (26) into $\left\langle\frac{1}{1+\tilde{\alpha}}\right\rangle$, we will have

$$\varepsilon^{-1}\left\langle\frac{1}{1+\tilde{\alpha}}\right\rangle^{-1}=1+\left\langle\frac{1}{1+\tilde{\alpha}}\right\rangle^{-1}\left\langle\frac{\tilde{q}_\delta}{1+\tilde{\alpha}}\right\rangle. \tag{27}$$

It is required to prove the inequality

$$\varepsilon\geq\left\langle\frac{1}{1+\tilde{\alpha}}\right\rangle^{-1} \tag{28}$$

or the equivalent inequality

$$\varepsilon^{-1}\left\langle\frac{1}{1+\tilde{\alpha}}\right\rangle^{-1}\leq 1. \tag{29}$$

Turning our attention to relation (27), we note that the satisfaction of the inequality

$$\left\langle\frac{1}{1+\tilde{\alpha}}\right\rangle^{-1}\left\langle\frac{\tilde{q}_\delta}{1+\tilde{\alpha}}\right\rangle\leq 0 \tag{30}$$

is the condition sufficient for the validity of Eq. (29).

Multiplication of both sides of Eq. (25) by $1+\tilde{q}_\delta$ and averaging yield

$$\varepsilon^{-1}\left(1+\langle\tilde{\vartheta}_\delta\tilde{q}_\delta\rangle\right)=\left\langle\frac{1}{1+\tilde{\alpha}}\right\rangle+2\left\langle\frac{\tilde{q}_\delta}{1+\tilde{\alpha}}\right\rangle+\left\langle\frac{\tilde{q}_\delta^2}{1+\tilde{\alpha}}\right\rangle, \tag{31}$$

or, with account for Eq. (26),

$$\varepsilon^{-1} = \frac{\left\langle \frac{1}{1+\tilde{\alpha}} \right\rangle - \left\langle \frac{\tilde{q}_\delta^2}{1+\tilde{\alpha}} \right\rangle}{1 - \langle \tilde{\vartheta}_\delta \tilde{q}_\delta \rangle} . \quad (32)$$

Having divided both sides of Eq. (32) into $\left\langle \frac{1}{1+\tilde{\alpha}} \right\rangle$, we obtain

$$\varepsilon^{-1} \left\langle \frac{1}{1+\tilde{\alpha}} \right\rangle^{-1} = \frac{1 - \left\langle \frac{1}{1+\tilde{\alpha}} \right\rangle^{-1} \left\langle \frac{\tilde{q}_\delta^2}{1+\tilde{\alpha}} \right\rangle}{1 - \langle \tilde{\vartheta}_\delta \tilde{q}_\delta \rangle} . \quad (33)$$

Since we always have $\left\langle \frac{1}{1+\tilde{\alpha}} \right\rangle \geq 0$ and $\left\langle \frac{\tilde{q}_\delta^2}{1+\tilde{\alpha}} \right\rangle \geq 0$, then $\left\langle \frac{1}{1+\tilde{\alpha}} \right\rangle^{-1} \left\langle \frac{\tilde{q}_\delta^2}{1+\tilde{\alpha}} \right\rangle \geq 0$. It follows that the satisfaction of inequality (7) (which was proved above) will be the condition needed to satisfy inequality (29).

Thus, from Eqs. (24) and (28) follows the validity of double inequality (6) that determines the range of variation in the conjugation parameter ε in the processes of heat exchange with periodic intensity.

NOTATION

τ, time; x and z, transverse and longitudinal coordinates; δ, wall thickness; λ, ρ, and c, thermal conductivity, density, and specific heat of the wall; α, true heat-transfer coefficient; α_m. experimental heat-transfer coefficient; ε, conjugation parameter; ϑ, temperature; q, heat-flux density; q_x and q_z, components of the heat-flux density along the coordinates x and z, respectively; $\langle\ \rangle$, averaging over time τ and over the coordinate z; tilde, dimensionless periodic component. Subscripts: δ, conditions with $x = \delta$; 0, conditions with $x = 0$.

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